European Journal of Combinatorics 51 (2016) 1–11

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Non-abelian almost totally branched coverings over the platonic maps

Kan Hu [a,b,d], Gareth A. Jones [c], Roman Nedela [d,e], Na-Er Wang [a,b,d]

[a] School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan, Zhejiang 316022, People's Republic of China

[b] Key Laboratory of Oceanographic Big Data Mining & Application of Zhejiang Province, Zhoushan, Zhejiang 316022, People's Republic of China

[c] School of Mathematics, University of Southampton, Southampton SO17 1BJ, United Kingdom

[d] Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01, Banská Bystrica, Slovak Republic

[e] Institute of Mathematics, Slovak Academy of Sciences, Banská Bystrica, Slovak Republic

ARTICLE INFO

Article history:
Received 11 July 2014
Accepted 27 April 2015
Available online 22 May 2015

ABSTRACT

A map is a 2-cell embedding of a connected graph into a closed surface. A map is orientable if the supporting surface is orientable. An orientable map is regular if its group of orientation-preserving automorphisms acts transitively on the darts. Using an equivalent algebraic description of regular maps and their coverings, we employ the theory of group extensions to classify the almost totally branched coverings of the platonic maps with non-abelian covering transformation groups, generalising the results of Hu, Nedela and Wang.

1. Introduction

A map $\mathcal{M}$ is an embedding $i : X \hookrightarrow \mathcal{S}$ of a connected graph X into a closed surface $\mathcal{S}$ such that each component of $\mathcal{S} \setminus i(X)$ is homeomorphic to an open disc. A map is orientable if its supporting surface $\mathcal{S}$ is orientable; otherwise, it is called non-orientable. Throughout the paper, maps considered are orientable. For simplicity, we also assume that they contain no semi-edges. An (orientation-preserving) automorphism of a map $\mathcal{M}$ is an automorphism of the underlying graph X, regarded as a

E-mail address: G.A.Jones@maths.soton.ac.uk (G.A. Jones).

http://dx.doi.org/10.1016/j.ejc.2015.04.008
0195-6698/

permutation of the darts preserving the graph incidence, which extends to an orientation-preserving homeomorphism of the supporting surface. It is well known that the group Aut $(\mathcal{M})$ of automorphisms has a semi-regular action on the darts of $\mathcal{M}$. If this action is transitive, and hence regular, then the map $\mathcal{M}$ is called regular.

The most prominent examples of regular maps are the *platonic* maps on the sphere. They have type $\{n, m\}$, meaning that all faces are n-gons and all vertices have valency m, where m and n are integers greater than 1 such that $1/m + 1/n > 1/2$. One of the nice features of the platonic maps is that there is only one regular map of each type.

As one of the central problems in topological graph theory, the classification of regular maps has been intensively investigated, typically by imposing certain conditions on the supporting surfaces, the embedded graphs or the underlying automorphism groups; see [18] for a survey. Another different approach to this problem is to construct and classify regular maps arising from regular coverings over a given regular map [5,6,9,11,13–17,19,20]. In this direction, regular coverings over the platonic maps have received particular attention. In the early stage, this was mostly investigated in the context of certain extensions of polyhedral groups [3,14,16]. In 1970s, Biggs [1] and Gross [4] independently developed a new approach to this problem under different names. The first is known as the homological method and the other is the voltage method. As pointed out by Surowski and Schroeder the second is just an "embryonic form" of the first [20].

The cyclic regular coverings of the platonic maps, branched exclusively over the face-centres or over the vertices, are classified in [11,19]. The self-dual cyclic regular coverings of the tetrahedral map, branched simultaneously over the face-centres and over the vertices, are classified in [17, Theorem 7]. The abelian regular coverings over the platonic maps have been studied by Jones in [9]. Recently, a complete classification of cyclic regular coverings over the platonic maps, possibly branched simultaneously over the face-centres and over the vertices, has been obtained in [6, Theorem 15].

In [6], the authors have indeed investigated a broader family of coverings called *almost totally branched coverings* between regular maps. They showed that the covering transformation group of an almost totally branched covering is a metacyclic group of rank at most 2, either abelian or non-abelian [6, Lemma 10]. In that paper, the abelian almost totally branched coverings of the platonic maps are classified [6, Theorem 15]. In this note, we present a classification of non-abelian almost totally branched coverings over the platonic maps.

2. Almost totally branched coverings

In this section, the algebraic theory of regular maps and coverings between them developed by Jones and Singerman in [10] is briefly outlined.

For a regular map $\mathcal{M}$ of type $\{n, m\}$, let x and y, respectively, generate the cyclic stabilisers of a vertex v and an edge e incident with v. Then $z = (xy)^{-1}$ generates the stabiliser of a face incident with both v and e, and $x^m = y^2 = z^n = xyz = 1$. It follows from the connectivity of the underlying graph of $\mathcal{M}$ that Aut $(\mathcal{M}) = \langle x, y\rangle$. Conversely, each generating pair (x, y) of a two-generated group G such that $y^2 = 1$ gives rise to a regular map $\mathcal{M}$ with Aut $(\mathcal{M}) \cong G$: We identify the darts of $\mathcal{M}$ with the elements of the group G, and in the left regular representation ρ of G, the cycles of ρ_x and ρ_y are identified with the vertices and edges of a connected graph X with incidence given by nonempty intersection. The successive powers of ρ_x give the local rotation of the darts around each vertex, and these local rotations determine an embedding of X into an oriented surface. The right regular representation of G is identified with the automorphism group of $\mathcal{M}$. Therefore, regular maps $\mathcal{M}$ correspond to *algebraic maps*, that is, triples (G, x, y) where $G = \langle x, y\rangle$ and $y^2 = 1$.

The topological notion of a "covering" between regular maps can be described algebraically. In particular, if a covering has branch points at vertices and at face-centres simultaneously, then it will be more convenient to replace the triple (G, x, y), $y^2 = 1$, with the triple (G, x, z) where $z = (xy)^{-1}$ so that $(xz)^2 = 1$. Let $\mathcal{N} = (G_1, x_1, z_1)$ and $\mathcal{M} = (G_2, x_2, z_2)$ be regular maps. We say that $\mathcal{N}$ is a (regular) *covering* of $\mathcal{M}$ if the assignment $\pi : x_1 \mapsto x_2, z_1 \mapsto z_2$ extends to an epimorphism from G_1 onto G_2. The kernel is called the *group of covering transformations* and is denoted by $CT(\pi)$. In particular, $\mathcal{N}$ is *isomorphic* to $\mathcal{M}$ if π extends to a group isomorphism. Therefore, for a given regular map $\mathcal{M} = (G_2, x_2, z_2)$, the determination of regular coverings $\mathcal{N} = (G_1, x_1, z_1)$ of $\mathcal{M}$ with a covering

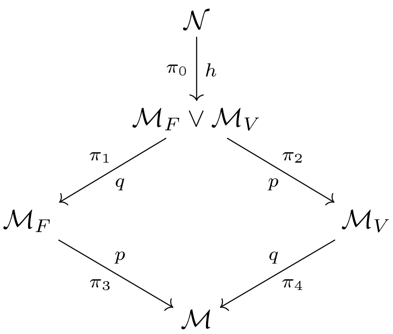

Fig. 1. Decomposition of an almost totally branched covering.

transformation group isomorphic to a group K is reduced to the determination of particular extensions of the group G_2 by K.

If $\mathcal{M} = (G, x, z)$ is a regular map, the mirror image of $\mathcal{M}$ is the regular map $\mathcal{M}^{-1} = (G, x^{-1}, z^{-1})$, and the dual of $\mathcal{M}$ is the map $\mathcal{M}^* = (G, z, x)$. Then $\mathcal{M}$ is reflexible if $\mathcal{M} \cong \mathcal{M}^{-1}$; otherwise, it is chiral; $\mathcal{M}$ is self-dual if $\mathcal{M} \cong \mathcal{M}^*$. It follows that reflexible (resp. self-dual) regular maps $\mathcal{M}$ with $\mathrm{Aut}\,(\mathcal{M}) \cong G$ correspond to *invertible* (resp. *transpositional*) generating pairs (x, z), that is, the assignment $\iota : x \mapsto x^{-1}, z \mapsto z^{-1}$ (resp. $\tau : x \mapsto z, z \mapsto x$) extends to an automorphism of G. For a given reflexible (resp. self-dual) regular map $\mathcal{M} = (G, x, z)$, if N is an ι-invariant (resp. τ-invariant) normal subgroup of G, then the quotient regular map $\bar{\mathcal{M}} = (G/N, xN, yN)$ is also reflexible (resp. self-dual) [7, Proposition 3].

Assume that the base map $\mathcal{M}$ has type $\{n, m\}$. Then the covering transformation group $CT(\pi)$ contains both $A = \langle x_1^m \rangle$ and $B = \langle z_1^n \rangle$. The covering is branched at vertices if and only if $A > 1$, and branched at faces if and only if $B > 1$. In the extremal cases that $CT(\pi) = A$ or $CT(\pi) = B$, the covering is called *totally branched at vertices* or *totally branched at faces*, respectively. Combinatorially speaking, the covering is totally branched at vertices (resp. at faces) if and only if every vertex (resp. every face) of $\mathcal{M}$ has precisely one preimage [6, Proposition 4]. A covering is called *totally branched* if it is both totally branched at vertices and totally branched at faces. More generally, the covering is called *almost totally branched* if $A \trianglelefteq G$, $B \trianglelefteq G$ and $CT(\pi) = AB$. Define

$$h = |A \cap B|, \qquad p = |A/A \cap B| \quad \text{and} \quad q = |B/A \cap B|. \tag{1}$$

Since $A, B \trianglelefteq G$ and $A \cap B = \langle z_1^{nq} \rangle = \langle x_1^{mp} \rangle$, there exist integers $e_V \in \mathbb{Z}_{ph}^*$, $e_F \in \mathbb{Z}_{qh}^*$ and $e \in \mathbb{Z}_h^*$ such that

$$(x_1^m)^{z_1} = (x_1^m)^{e_V}, \qquad (z_1^n)^{x_1} = (z_1^n)^{e_F} \quad \text{and} \quad z_1^{nq} = (x_1^{mp})^e. \tag{2}$$

The 6-tuple (p, q, h, e_V, e_F, e) associated with an almost totally branched covering is called the *index-exponent 6-tuple* of the covering. Note that the number ph (resp. qh) is the branch index of the covering at branch points over vertices (resp. face-centres).

An almost totally branched covering $\pi : \mathcal{N} \to \mathcal{M}$ between two regular maps is a composition of several extremal subcoverings [6, Proposition 7]. More precisely, let $\mathcal{M}_F = \mathcal{N}/B$, $\mathcal{M}_V = \mathcal{N}/A$ and $\mathcal{M}_F \vee \mathcal{M}_V = \mathcal{N}/A \cap B$. Then the covering $\pi_0 : \mathcal{N} \to \mathcal{M}_F \vee \mathcal{M}_V$ is totally branched, the coverings $\pi_1 : \mathcal{M}_F \vee \mathcal{M}_V \to \mathcal{M}_F$ and $\pi_4 : \mathcal{M}_V \to \mathcal{M}$ are totally branched at faces and smooth at vertices, the coverings $\pi_2 : \mathcal{M}_F \vee \mathcal{M}_V \to \mathcal{M}_V$ and $\pi_3 : \mathcal{M}_F \to \mathcal{M}$ are totally branched at vertices and smooth at faces. In particular, the diagram in Fig. 1 commutes.

In what follows, let $X^{(m)}$ denote a graph of multiplicity m obtained from a simple graph X by replacing each of the edges with m parallel edges.

Example 1. In the simplest case, $\mathcal{M}$ is the tetrahedral map $\mathcal{T}$ of type $\{3, 3\}$ with the complete graph K_4 of order 4 as its underlying graph. The 2-sheeted regular covering of $\mathcal{T}$ of type $\{6, 3\}$, totally branched at faces and smooth at vertices, is a regular embedding $\mathcal{Q} = \mathcal{M}_V$ of the 3-dimensional cube Q_3 into the torus [11, Theorem 1]. Since $\mathcal{T}$ is self-dual, the dual map $\mathcal{Q}^* = \mathcal{M}_F$ of $\mathcal{Q}$ is a 2-sheeted regular

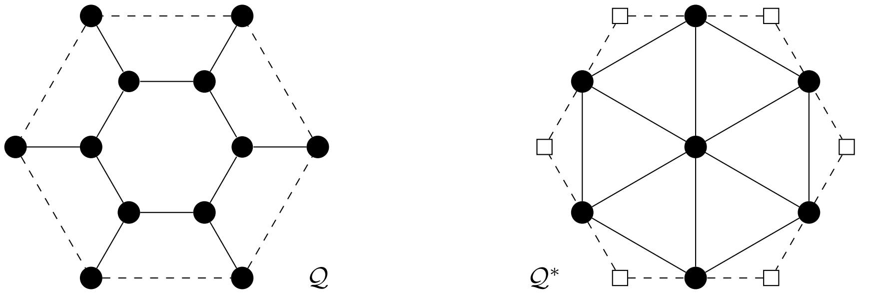

Fig. 2. The torus maps $\mathcal{Q}$ and $\mathcal{Q}^*$.

covering of $\mathcal{T}$ of genus 1 and of type $\{3, 6\}$, totally branched at vertices and smooth at faces. It is a regular embedding of the graph $K_4^{(2)}$. These maps are shown in Fig. 2; in each case opposite sides of the outer hexagon are identified to form a torus, and the covering of $\mathcal{T}$ is induced by a half-turn about the centre of the hexagon.

The join $\mathcal{Q} \vee \mathcal{Q}^*$ is the smallest regular map covering both $\mathcal{Q}$ and $\mathcal{Q}^*$. This is a regular covering of $\mathcal{T}$, of genus 5, appearing as entry R5.10 in Conder's list of regular maps [2]. It has underlying graph $Q_3^{(2)}$ and automorphism group given by the presentation

$$\langle x, z \mid x^6 = (xz)^2 = z^6 = [x^3, z] = [x, z^3] = 1\rangle.$$

This map is a 4-sheeted abelian almost totally branched covering of $\mathcal{T}$ with the covering transformation group given by $\langle x^3, z^3\rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Further, we may construct two non-isomorphic 2-sheeted totally branched coverings $\mathcal{N}(r)$ $(r = 1, 3)$ over the map $\mathcal{Q} \vee \mathcal{Q}^*$ with their automorphism groups given by the presentation

$$\langle x, z \mid x^{12} = (xz)^2 = 1, (x^3)^z = x^{3r}, (z^3)^x = z^{3r}, z^6 = x^6\rangle.$$

These are regular embeddings of $Q_3^{(4)}$, corresponding to Conder's maps R17.34 and R17.35 [2]. It is easily seen that both $\mathcal{N}(1) \to \mathcal{T}$ and $\mathcal{N}(3) \to \mathcal{T}$ are almost totally branched coverings, with the covering transformation groups isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$ and Q_8 respectively.

The following lemma describes the structure of the covering transformation group of an almost totally branched covering; see [6] for its proof.

Lemma 1 ([6, Lemma 10]). *Let $\mathcal{M}$ be a regular map of type $\{n, m\}$, and let $\pi : \mathcal{N} \to \mathcal{M}$ be an almost totally branched covering of $\mathcal{M}$ where $\mathcal{N} = (G, x, z)$, with the associated index-exponent 6-tuple (p, q, h, e_V, e_F, e) where $\gcd(e, h) = 1$; let $a = x^m$ and $b = z^n$. If the covering transformation group $CT(\pi)$ is non-abelian, then both m and n are odd, p, q and h are all even,*

$$e_F \equiv 1 + \frac{qh}{2} \pmod{qh} \quad \text{and} \quad e_V \equiv 1 + \frac{ph}{2} \pmod{ph}, \tag{3}$$

and $CT(\pi)$ has a presentation

$$\langle a, b \mid a^{ph} = b^{qh} = 1, b^a = b^{1+\frac{qh}{2}}, a^b = a^{1+\frac{ph}{2}}, b^q = a^{pe}\rangle. \tag{4}$$

Remark 1. Let $\pi : \mathcal{N} \to \mathcal{M}$ be an almost totally branched covering between two regular maps where $\mathcal{N} = (G, x, z)$. If the underlying graphs of $\mathcal{M}$ and its dual $\mathcal{M}^*$ are both simple graphs, then $A = \mathrm{core}_G\langle x\rangle$ and $B = \mathrm{core}_G\langle z\rangle$, namely the cores of $\langle x\rangle$ and $\langle z\rangle$ in $G = \mathrm{Aut}(\mathcal{N})$ [12]. It follows that the underlying graphs of the maps $\mathcal{N}$ and its dual $\mathcal{N}^*$ have multiplicity $m_V = |A|$ and $m_F = |B|$ [6, Proposition 4]. The pair (m_F, m_V) is called the *multiplicity type* of $\mathcal{N}$, or simply M-type of $\mathcal{N}$.

The following lemma deals with the reflexibility and self-duality of the regular maps in an almost totally branched covering.

Lemma 2. *Let $\mathcal{M}$ be a regular map of type $\{n, m\}$, and let $\mathcal{N} \to \mathcal{M}$ be an almost totally branched covering of $\mathcal{M}$ with an index-exponent* 6*-tuple* (p, q, h, e_V, e_F, e).

(i) *If $\mathcal{N}$ is reflexible, then so is $\mathcal{M}$;*
(ii) *Assume that both $\mathcal{M}$ and its dual map $\mathcal{M}^*$ have simple underlying graphs. If $\mathcal{N}$ is self-dual, then so is $\mathcal{M}$. In particular,*

$$m = n, \qquad p = q, \qquad e_V = e_F \quad \textit{and} \quad e^2 \equiv 1 \pmod{h}.$$

Proof. Let $\mathcal{N} = (G_1, x_1, z_1)$ and $K = \langle x_1^m, z_1^n \rangle$. Then K is the covering transformation group with the relations in (2) satisfied by x_1 and z_1. Let $w = w(x_1^m, z_1^n)$ be any word in K. Since K is metacyclic, there exist some integers i and j such that $w = (x_1^m)^i (z_1^n)^j$.

(i) We have $\iota(w) = (x_1^{-m})^i (z_1^{-n})^j \in K$. Hence $K^\iota = K$, that is, K is an ι-invariant normal subgroup of G_1. Consequently, since $\mathcal{M} \cong \mathcal{N}/K$ and $\mathcal{N}$ is reflexible, $\mathcal{M}$ is also reflexible.

(ii) By the assumption, both $\mathcal{M}$ and its dual $\mathcal{M}^*$ have simple underlying graphs. By Remark 1, $\mathrm{Core}_{G_1}\langle x_1 \rangle = \langle x_1^m \rangle$ and $\mathrm{Core}_{G_1}\langle z_1 \rangle = \langle z_1^n \rangle$. Since $\mathcal{N}$ is self-dual, G_1 has an automorphism τ transposing x_1 and z_1, so $m = n$ and $o(x_1) = o(z_1)$. We have $\tau(w) = \tau((x_1^m)^i (z_1^m)^j) = (z_1^m)^i (x_1^m)^j \in K$. So K is a τ-invariant normal subgroup of G_1. Consequently, since $\mathcal{M} = \mathcal{N}/K$ and $\mathcal{N}$ is self-dual, $\mathcal{M}$ is also self-dual.

Recall that $o(x_1) = mph$ and $o(z_1) = nqh$. Since $o(x_1) = o(z_1)$, we have $p = q$. By comparing the first two relations in (2) we have $e_V = e_F$. Moreover, by applying τ to the last relation in (2) we obtain $x_1^{mp} = z_1^{mpe}$. Substituting x_1^{mpe} for z_1^{mp} we get $x_1^{mp} = z_1^{mpe} = x_1^{mpe^2}$. Hence $x_1^{mp(e^2-1)} = 1$. Since $o(x_1) = mph$, we get $e^2 \equiv 1 \pmod{h}$, as required. $\square$

3. Classification

In this section, we employ the theory of group extensions to classify the non-abelian almost totally branched coverings over the platonic maps.

Recall that the platonic maps are regular maps of types $\{n, m\}$ where $m, n \geq 2$ such that $1/n + 1/m > 1/2$. Their data are summarised in Table 1.

Table 1
The platonic maps.

Map	Type	M-Type	$\|V\|$	$\|F\|$	Aut.
Tetrahedral map	$\{3, 3\}$	$(1, 1)$	4	4	A_4
Cube	$\{4, 3\}$	$(1, 1)$	8	6	S_4
Octahedral map	$\{3, 4\}$	$(1, 1)$	6	8	S_4
Icosahedral map	$\{5, 3\}$	$(1, 1)$	12	20	A_5
Dodecahedral map	$\{3, 5\}$	$(1, 1)$	20	12	A_5
Dihedral map	$\{n, 2\}$	$(n, 1)$	n	2	D_{2n}
Hosohedral map	$\{2, n\}$	$(1, n)$	2	n	D_{2n}

The following proposition on cyclic extensions of groups will be useful.

Proposition 3 ([8, *Theorem 3.36*]). *Let K and Q be groups, where Q is cyclic of order m, let $a \in K$ and $\sigma \in \mathrm{Aut}(K)$. Assume that*

$$a^\sigma = a \quad \textit{and} \quad x^{\sigma^m} = x^a, \ \textit{for all } x \in K.$$

Then there exist an extension G of Q by K, unique up to isomorphism, and an element $g \in G$ with the following properties:

(i) $G/K = \langle gK \rangle \cong Q$,
(ii) $g^m = a$,
(iii) $x^\sigma = x^g$.

The following theorem classifies the non-abelian almost totally branched coverings over the platonic maps.

Theorem 4. *In the family of platonic maps, only the tetrahedral map, the icosahedral map and the dodecahedral map admit non-abelian almost totally branched coverings. Moreover, the isomorphism classes of non-abelian almost totally branched coverings $\mathcal{N}$ over an admissible platonic map $\mathcal{M}$ of type $\{n, m\}$ are in one-to-one correspondence with the quadruples (p, q, h, e) of positive integers satisfying the following conditions:*

(i) *p, q and h are all even,*
(ii) *p divides $|V|$ and q divides $|F|$, where $|V|$ and $|F|$ denote the numbers of vertices and faces of $\mathcal{M}$,*
(iii) $\frac{|F|}{q}e + \frac{|V|}{p} \equiv 0 \pmod h$ *and* $e \in \mathbb{Z}_h^*$.

The group Aut $(\mathcal{N})$ *has a presentation*

$$\langle x, z \mid x^{mph} = (xz)^2 = z^{nqh} = 1, (x^m)^z = x^{m(1+ph/2)}, (z^n)^x = z^{n(1+qh/2)}, z^{nq} = x^{mpe}\rangle \quad (5)$$

with Aut $(\mathcal{N})/K \cong$ Aut $(\mathcal{M})$ *where* $K = \langle x^m, z^n\rangle$.

In particular, all such covers are reflexible, and the covers over the tetrahedral map are all self-dual, whereas none of the covers over the dodecahedral map or over the icosahedral map are self-dual.

Proof. By Lemma 1, if $\mathcal{M}$ admits a non-abelian almost totally branched cover $\mathcal{N}$, then both n and m are odd. Checking the types of the platonic maps, we see that only the tetrahedral map, the icosahedral map and the dodecahedral map may admit non-abelian almost totally branched coverings. Therefore, by duality, it is sufficient to classify such covers over the tetrahedral map and the icosahedral map.

Let $\mathcal{M} = (G_2, x_2, z_2)$, $\mathcal{N} = (G, x, z)$, $a = x^m$ and $b = z^n$. Since the covering is almost totally branched, the assignment $x \mapsto x_2, z \mapsto z_2$ extends to an epimorphism from G onto G_2 with a covering transformation group $K := AB$ where $A = \langle a\rangle$ and $B = \langle b\rangle$ are normal subgroups of G. By Lemma 1, p, q and h are even, $e_F = 1 + qh/2$, $e_V = 1 + ph/2$ and $\gcd(e, h) = 1$, and the covering transformation group K has a presentation (4) where the integers p, q, h, e_F, e_V and e are defined by (1) and (2). Therefore, by the theory of group extensions, the group G has a presentation (5).

In what follows, for each case we shall construct a chain of subnormal subgroups of G and then employ the theory of cyclic extensions of groups to derive the numerical conditions (ii) and (iii) stated in the theorem.

Case (1). Let the base map be the tetrahedral map of type $\{3, 3\}$.

Let $c_1 = z^2x^2$ and $c_2 = zx$, and define $N_1 = \langle K, c_1\rangle$ and $N_2 = \langle N_1, c_2\rangle$. Then $G = \langle N_2, x\rangle$. Since $x^3 = a$, $c_2^2 = (zx)^2 = 1$,

$$c_1^2 = (z^2x^2)^2 = (z^3z^{-1}x^{-1}x^3)^2 = (z^3x^{3e_V}z^{-1}x^{-1})^2 = b^{1+e_F}a^2 \quad (6)$$

and

$$c_1^{c_2} = x^{-1}zx^2zx = x^{-1}zxz^{-1} = x^{-2}z^{-2} = c_1^{-1}, \quad (7)$$
$$c_1^x = x^{-1}z^2x^3 = x^{-1}z^{-1}ba = c_2^{-1}ba, \quad (8)$$
$$c_2^x = x^{-1}zx^2 = x^{-1}z^{-1}(z^2x^2) = c_2^{-1}c_1, \quad (9)$$

G has a chain of subnormal subgroups $1 \trianglelefteq K \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq G$ such that

$$N_1/K = \langle c_1K\rangle \cong \mathbb{Z}_2, \qquad N_2/N_1 = \langle c_2N_1\rangle \cong \mathbb{Z}_2, \qquad G/N_2 = \langle xN_2\rangle \cong \mathbb{Z}_3.$$

By (7) and (6), we have

$$(c_1^{c_2})^2 = c_1^{-2} = b^{-(1+e_F)}a^{-2}$$

and

$$(c_1^2)^{c_2} = (b^{1+e_F}a^2)^{c_2} = (b^{c_2})^{1+e_F}(a^{c_2})^2 = b^{1+e_F}a^{2e_V}.$$

Since $(c_1^{c_2})^2 = (c_1^2)^{c_2}$, by equating the right hand sides of the above equations, we obtain that $b^{2(1+e_F)} = a^{-2(1+e_V)}$. Recall that $e_F = 1 + qh/2$ and $e_V = 1 + ph/2$. Upon substitution we get

$$b^{4+qh} = a^{-4-ph}. \quad (10)$$

Since $\langle a\rangle \cap \langle b\rangle = \langle b^q\rangle = \langle a^p\rangle$, we obtain

$$4 \equiv 0 \pmod q \quad \text{and} \quad 4 \equiv 0 \pmod p. \tag{11}$$

Recall that $b^q = a^{pe}$. Upon substitution (10) is reduced to

$$a^{-4-ph} = b^{4+qh} = (b^q)^{4/q+h} = (a^{pe})^{4/q+h}.$$

Consequently $a^{p(4/p+4e/q+(e+1)h)} = 1$. Since a has order ph, we obtain

$$4e/q + 4/p \equiv 0 \pmod h. \tag{12}$$

Note that the group G has an alternative presentation

$$\begin{aligned} G = \langle a, b, c_1, c_2, x \mid\ & a^{ph} = b^{qh} = 1, b^a = b^{1+qh/2}, a^b = a^{1+ph/2},\\ & b^q = a^{pe}, c_1^2 = b^{2+qh/2}a^2, b^{c_1} = b, a^{c_1} = a, c_2^2 = 1, a^{c_2} = a^{1+ph/2},\\ & b^{c_2} = b^{1+qh/2}, c_1^{c_2} = c_1^{-1}, x^3 = a, a^x = a, b^x = b^{1+qh/2},\\ & c_1^x = ab^{1+qh/2}c_2^{-1}, c_2^x = c_2^{-1}c_1\rangle. \end{aligned} \tag{13}$$

Conversely, given a group G defined by (5) where $m = n = 3$ and p, q and h are positive even integers, $e \in \mathbb{Z}_h^*$, and they satisfy the conditions (11) and (12), by applying Proposition 3 it is straightforward to verify that G is a well-defined extension of the group G_2 by K. In particular, $|G| = 3|N_2| = 6|N_1| = 12|K| = 12pqh$. It is easily seen from the presentation that the map (G, x, z) is a non-abelian almost totally branched cover over the tetrahedral map with an index-exponent 6-tuple $(p, q, h, 1 + ph/2, 1 + qh/2, e)$.

Case (2). Let the base map be the icosahedral map of type $\{3, 5\}$.

Let $c_1 = (xz)^{x^{-2}}(xz)^{x^{-1}}$ and $c_2 = (xz)^{x^{-1}}$, and define $N_1 = \langle K, c_1\rangle$ and $N_2 = \langle N_1, c_2\rangle$. We deduce from (2) that

$$b^{c_1} = b, \qquad a^{c_1} = a, \qquad b^{c_2} = b^{e_F} \quad \text{and} \quad a^{c_2} = a^{e_V}. \tag{14}$$

Note that using the commuting rules in (2) and the fact that $z^{-1}x^{-1} = xz$ the element c_1 can be simplified as follows:

$$c_1 = x^3z^2x^{-1} = x^3b(z^{-1}x^{-1}) = b^{e_F}x^4z = b^{e_F}ax^{-1}z \tag{15}$$

$$= b^{e_F}a(x^{-1}z^{-1})z^{-1}b = b^{1+e_F}a(zxz^{-1}). \tag{16}$$

We have

$$\begin{aligned} c_1^{c_2} &= xz^{-2}x^{-3} = xzz^{-3}x^{-3}\\ &= b^{-e_F}xzx^{-3} = b^{-e_F}z^{-1}xx^{-5}\\ &= b^{-e_F}a^{-e_V}z^{-1}x\\ &= b^{-e_F}a^{-e_V}c_1^{-1}b^{e_F}a = c_1^{-1} \qquad \text{by (15)} \end{aligned} \tag{17}$$

and

$$c_1^5 = (b^{1+e_F}azxz^{-1})^5 = b^{5(1+e_F)}a^{5+e_V}. \qquad \text{by (16)} \tag{18}$$

Combining these with (14) we see that the subgroup N_2 has a chain of normal subgroups $1 \trianglelefteq K \trianglelefteq N_1 \trianglelefteq N_2$ such that

$$N_1/K = \langle c_1K\rangle \cong \mathbb{Z}_5, \qquad N_2/N_1 = \langle c_2N_1\rangle \cong \mathbb{Z}_2.$$

By (17) and (18) and using (14) when necessary we deduce that

$$(c_1^5)^{c_2} = (b^{5(1+e_F)}a^{5+e_V})^{c_2} = b^{5(1+e_F)}a^{5e_V+1}$$

and

$$(c_1^{c_2})^5 = c_1^{-5} = a^{-5-e_V}b^{-5(1+e_F)} \overset{(2)}{=} b^{-5(1+e_F)}a^{-5-e_V}.$$

Since $(c_1^5)^{c_2} = (c_1^{c_2})^5$, we get $a^{6(e_V+1)} = b^{-10(e_F+1)}$. Recall that $e_F = 1 + qh/2$ and $e_V = 1 + ph/2$. Upon substitution we obtain

$$a^{12} = b^{-20}. \tag{19}$$

Recall that $\langle a\rangle \cap \langle b\rangle = \langle b^q\rangle = \langle a^p\rangle$. We deduce from (19) that

$$20 \equiv 0 \pmod q \quad \text{and} \quad 12 \equiv 0 \pmod p. \tag{20}$$

Since $b^q = a^{pe}$, substituting b^q for a^{pe} in (19) we obtain

$$a^{12} = b^{-20} = (b^q)^{-20/q} = a^{-20ep/q}.$$

Hence $a^{p(20e/q+12/p)} = 1$. Since a has order ph, we get

$$20e/q + 12/p \equiv 0 \pmod h. \tag{21}$$

Let H be the group defined by the presentation

$$\langle x, z \mid (xz)^2 = z^{3qh} = x^{5ph} = 1, (z^3)^x = z^{3(1+qh/2)}, (x^5)^z = x^{5(1+ph/2)}, z^{3q} = x^{5pe}\rangle, \tag{22}$$

where the parameters p, q and h are even, $\gcd(e, h) = 1$ and they satisfy (20) and (21). Clearly, G satisfies all the defining relations of H. It follows that G is a quotient of H. Define $T = \langle x^5, z^3\rangle$. We see from the presentation of H that $T \trianglelefteq H$ and $H/T \cong A_5$, so $|H| = |T||A_5| \le |G|$. Therefore $G \cong H$. Note that the above discussion shows that N_2 is a subgroup of G of index 6 constructed by a sequence of cyclic extensions. It has a presentation

$$\begin{aligned} N_2 = \langle a, b, c_1, c_2 \mid\ & a^{ph} = b^{qh} = 1, a^b = a^{1+ph/2}, b^a = b^{1+qh/2}, b^q = a^{pe}, \\ & c_1^5 = b^{5(2+ph/2)}a^{5(1+ph/2)+1}, a^{c_1} = a, b^{c_1} = b, \\ & c_2^2 = 1, a^{c_2} = a^{1+ph/2}, b^{c_2} = b^{1+qh/2}, c_1^{c_2} = b^{-qh/2}a^{-ph/2}c_1^{-1}\rangle. \end{aligned} \tag{23}$$

Conversely, given a group G defined by (22) with the stated numerical conditions satisfied by the parameters, to show that G is a well-defined extension of $G_2 \cong A_5$ by K we need to show that $|G| = 60qph$. By applying Proposition 3 it is straightforward to verify that the group N_2 given by (23) is a subgroup of G of order $10qph$. It is therefore sufficient to prove that $[G : N_2] = 6$. It is clear that $x^i \notin N_2$ $(i = 1, 2, 3, 4)$ and $z, z^{-1} \notin N_2$. We deduce from the presentation of G that $x^{-1}z = a^{-1}b^{-e_F}c_1 \in N_2$ (cf. (15)). Hence $xN_2 = zN_2$. Moreover, if $z^{-1}N_2 = x^iN_2$ for some i, $1 \le i \le 4$, then $zx^i \in N_2$. Since $x^{-1}z \in N_2$, we have $(x^{-1}z)(zx^i) \in N_2$. But

$$(x^{-1}z)(zx^i)N_2 = x^{-1}z^{-1}z^3x^iN_2 = zx^{i+1}(x^{-i}bx^i)N_2 = zx^{i+1}N_2,$$

so $zx^{i+1} \in N_2$. By induction, we deduce that $z \in N_2$, a contradiction. Therefore,

$$N_2, xN_2, x^2N_2, x^3N_2, x^4N_2, z^{-1}N_2$$

are distinct left cosets of N_2 in G. Consequently $[G : N_2] = 6$. Moreover, it is easily seen that the regular map (G, x, z) is a non-abelian almost totally branched cover over the icosahedral map. This deals with case (2).

In case (1) and case (2), let (p_i, q_i, h_i, e_i) be the quadruples corresponding to two coverings $\mathcal{N}_i$ over a common base map where the groups $\mathrm{Aut}\,(\mathcal{N}_i)$ $(i = 1, 2)$ are given by the presentation (5). Using the fact that group isomorphisms preserve defining relations, we deduce that $\mathcal{N}_1 \cong \mathcal{N}_2$ if and only if $p_1 = p_2, q_1 = q_2, h_1 = h_2$ and $e_1 \equiv e_2 \pmod{h_1}$. In particular, for each such cover $\mathcal{N}$, since the assignment $x \mapsto x^{-1}, z \mapsto z^{-1}$ extends to an automorphism of G, the map $\mathcal{N}$ is reflexible.

Finally, if $\mathcal{N}$ is self-dual, then by Lemma 2 the base map $\mathcal{M}$ is self-dual as well, implying that $\mathcal{M}$ is the tetrahedral map. In this case, by Lemma 2, we have $p = q$ and $e^2 \equiv 1 \pmod h$. Conversely, it is straightforward to verify that if $p = q$ and $e^2 \equiv 1 \pmod h$, then the non-abelian almost totally branched covering $\mathcal{N}$ over the tetrahedral map is a self-dual map. Recall that p, q and h are even. By (11), $p, q \in \{2, 4\}$. If $p = q = 4$, then $e = h - 1$; if $p = q = 2$ and $h \equiv 0 \pmod 4$, then $e = h/2 - 1$ or $e = h - 1$; if $p = q = 2$ and $h \equiv 2 \pmod 4$, then $e = h - 1$. In each case, $e^2 \equiv 1 \pmod h$. Therefore the coverings are all self-dual, as claimed. □

The following corollary follows from the Euler–Poincaré formula and [6, Proposition 4] (see also Remark 1).

Corollary 5. *The map $\mathcal{N}$ in Theorem 4 has type $\{nqh, mph\}$, multiplicity type (qh, ph) and genus*

$$g = \frac{mnpqh - 2nq - 2mp}{2m + 2n - mn} + 1.$$

4. Enumeration

Let $\mathcal{M}$ represent a fixed admissible platonic map, that is, $\mathcal{M}$ is the tetrahedral map, the icosahedral map, or the dodecahedral map, and let $\mathcal{N}$ be a non-abelian almost totally branched covering of $\mathcal{M}$. By Corollary 5, the genus g of $\mathcal{N}$ depends only on the variables p, q and h. Therefore, the number μ of maps $\mathcal{N}$ with a fixed triple (p, q, h) is equal to the number of values of e satisfying Theorem 4(iii). In this section, we first determine the admissible values of the triples (p, q, h), and then compute the number μ.

Before proceeding we prove a number-theoretic result for future reference.

Lemma 6. *Let p be a prime. Then the linear congruence*

$$ax \equiv b \pmod{p^e} \tag{24}$$

is solvable in $\mathbb{Z}^*_{p^e}$ if and only if $\gcd(a, p^e) = \gcd(b, p^e)$, in which case the number of solutions is $\varphi(p^e)/\varphi(p^e/\gcd(a, p^e))$ where φ is the Euler's totient function.

Proof. Let $\gcd(a, p^e) = p^d$ and $\gcd(b, p^e) = p^{d'}$. If (24) has a solution x_0 where $\gcd(x_0, p) = 1$, then $d = d'$. Conversely, if $d = d'$, then (24) is reduced to

$$ux \equiv v \pmod{p^{e-d}}, \tag{25}$$

where $u := a/p^d$ and $v := b/p^d$ are both invertible in $\mathbb{Z}_{p^{e-d}}$. Then $x_0 = v/u$ is the unique invertible solution of the congruence (25). It lifts to p^d invertible solutions of the form $x_0 + ip^{e-d}$ $(0 \le i \le p^d - 1)$ of the congruence (24) in $\mathbb{Z}_{p^e}$. □

Theorem 7. *Let a and b be integers, and let n be a positive integer. Then the linear congruence*

$$ax \equiv b \pmod{n} \tag{26}$$

is solvable in $\mathbb{Z}^*_n$ if and only if $\gcd(a, n) = \gcd(b, n)$, in which case the number of solutions is $\varphi(n)/\varphi(n/c)$, where $c = \gcd(a, n)$ and φ is the Euler's totient function.

Proof. Let $n = \prod_{i=1}^r p_r^{e_r}$ be the prime power factorisation of n. By the Chinese Remainder Theorem, (26) is solvable in $\mathbb{Z}^*_n$ if and only if the congruence $ax \equiv b \pmod{p_i^{e_i}}$ is solvable in $\mathbb{Z}^*_{p_i^{e_i}}$ for each p_i $(1 \le i \le r)$. By Lemma 6, this is equivalent to that $\gcd(a, p_i^{e_i}) = \gcd(b, p_i^{e_i})$. Note that $\gcd(a, n) = \prod_{i=1}^r \gcd(a, p_i^{e_i})$ and $\gcd(b, n) = \prod_{i=1}^r \gcd(b, p_i^{e_i})$. Therefore (26) is solvable in $\mathbb{Z}^*_n$ if and only if $\gcd(a, n) = \gcd(b, n)$, in which case, by Lemma 6, it has

$$\prod_{i=1}^r \frac{\varphi(p^{e_i})}{\varphi(p^{e_i}/\gcd(a, p^{e_i}))} = \frac{\varphi(\prod_{i=1}^r p^{e_i})}{\varphi(\prod_{i=1}^r p^{e_i}/\prod_{i=1}^r \gcd(a, p^{e_i}))} = \frac{\varphi(n)}{\varphi(n/c)}$$

solutions, as claimed. □

The following theorem determines the quadruples (p, q, h, e) corresponding to non-abelian almost totally branched coverings over an admissible platonic map $\mathcal{M}$. By Theorem 4 and duality, it is sufficient to state the result when $\mathcal{M}$ is the tetrahedral map or the icosahedral map.

Table 2
Almost totally branched covers over $\mathcal{T}$.

q	p	$h \in \mathbb{Z}^+$	$e \in \mathbb{Z}_h^*$	g	μ
2	2	2 (mod 4)	$h-1$	$12h-7$	1
2	2	0 (mod 4)	$h/2-1, h-1$	$12h-7$	2
4	4	0 (mod 2)	$h-1$	$48h-15$	1

Table 3
Almost totally branched covers over $\mathcal{I}$.

q	p	$h \in \mathbb{Z}^+$	$e \in \mathbb{Z}_h^*$	g	μ
2	2	$\pm 2, \pm 14 \pm 34, \pm 38$ (mod 60)	$5e+3 \equiv 0$ (mod $h/2$)	$60h-31$	1
2	2	$\pm 4, \pm 8 \pm 16, \pm 32$ (mod 60)	$5e+3 \equiv 0$ (mod $h/2$)	$60h-31$	2
2	6	$\pm 2, \pm 6$ (mod 20)	$5e+1 \equiv 0$ (mod $h/2$)	$180h-71$	1
2	6	$\pm 4, \pm 8$ (mod 20)	$5e+1 \equiv 0$ (mod $h/2$)	$180h-71$	2
4	4	$\pm 2, \pm 4, \pm 8, \pm 14$ (mod 30)	$5e+3 \equiv 0$ (mod h)	$240h-63$	1
4	12	$\pm 2, \pm 4$ (mod 10)	$5e+1 \equiv 0$ (mod h)	$720h-141$	1
10	2	± 2 (mod 12)	$h-3$	$300h-79$	1
10	2	± 4 (mod 12)	$h-3, h/2-3$	$300h-79$	2
10	6	2 (mod 4)	$h-1$	$900h-119$	1
10	6	0 (mod 4)	$h-1, h/2-1$	$900h-119$	2
20	4	± 2 (mod 6)	$h-3$	$1200h-159$	1
20	12	0 (mod 2)	$h-1$	$3600h-239$	1

Theorem 8. *The quadruples (p, q, h, e) corresponding to non-abelian almost totally branched coverings $\mathcal{N}$ over the tetrahedral map $\mathcal{T}$ or the icosahedral map $\mathcal{I}$ are summarised in Tables* 2 and 3*, where g denotes the genus of $\mathcal{N}$ and μ is the number of maps $\mathcal{N}$ with a given triple (p, q, h).*

Proof. First let $\mathcal{M} = \mathcal{T}$. By Theorem 4, the cover $\mathcal{N}$ over $\mathcal{M}$ corresponds to a quadruple (p, q, h, e) where p, q and h are even positive numbers, $e \in \mathbb{Z}_h^*$ and these parameters satisfy the congruences (11) and (12). By Theorem 4(i)–(iii), we have either $q = p = 2$ or $q = p = 4$. In the first case, (12) is reduced to

$$2(e+1) \equiv 0 \pmod{h}. \tag{27}$$

By Theorem 7, (27) has $\varphi(h)/\varphi(h/2)$ solutions $e \in \mathbb{Z}_h^*$. It is easily seen that $\varphi(h)/\varphi(h/2) = 1$ if $h \equiv 2 \pmod 4$, and $\varphi(h)/\varphi(h/2) = 2$ if $h \equiv 0 \pmod 4$. Similarly, in the other case, (12) is reduced to

$$e + 1 \equiv 0 \pmod{h},$$

which has a unique solution $e = h - 1$ in $\mathbb{Z}_h^*$.

Now let $\mathcal{M} = \mathcal{I}$. By Theorem 4(i)–(ii), the pair (q, p) takes the value (2, 2), (2, 6), (4, 4), (4, 12), (10, 2), (10, 6), (20, 4) or (20, 12). For each such pair, by applying Theorem 7 we obtain the values of h, and the number of solutions $e \in \mathbb{Z}_h^*$ of (21). □

Remark 2. By Theorem 8, the non-abelian almost totally branched coverings over the tetrahedral map have $4q$ vertices and $4p$ faces where $p, q \in \{2, 4\}$, and those over the icosahedral map have $12q$ vertices and $20p$ faces where $p \in \{2, 4, 6, 12\}$ and $q \in \{2, 4, 10, 20\}$. Therefore, though the number of non-abelian almost totally branched coverings $\mathcal{N}$ over each admissible platonic map is infinite, both the number of vertices and the number of faces of $\mathcal{N}$ are bounded above. This is also true for the abelian almost totally branched coverings over the same maps [6, Theorem 15].

Acknowledgements

This work is supported by the following grants: Scientific Research Foundation (SRF) of Zhejiang Ocean University (21065014015, 21065014115), "Mobility-Enhancing Research, Science and Education" at Matej Bel University in Banská Bystrica, ITMS code 26110230082, under the Operational Programme Education co-financed by the European Social Fund, APVV-0223-10, VEGA 1/1085/11, and from the grant APVV-ESF-EC-0009-10 within the EUROCORES Programme EUROGIGA (project GReGAS) of the European Science Foundation.

References

[1] N. Biggs, The symplectic representation of map automorphisms, Bull. Lond. Math. Soc. 4 (1972) 303–306.
[2] M.D.E. Conder, Reflexible orientable regular maps on surfaces of genus 2 to 301, online. August 2012. URL: http://www.math.auckland.ac.nz/~conder/OrientableRegularMaps301.txt.
[3] H.S.M. Coxeter, W.O.J. Moser, Generators and Relations for Discrete Groups, Springer-Verlag, Berlin, 1958.
[4] J.L. Gross, Voltage graphs, Discrete Math. 9 (1974) 239–246.
[5] K. Hu, Group extensions and coverings of maps (Ph.D. dissertation), Matej Bel University, Banská Bystrica, 2013.
[6] K. Hu, R. Nedela, N.-E. Wang, Branched cyclic regular coverings over platonic maps, European J. Combin. 36 (2014) 531–549.
[7] K. Hu, R. Nedela, N.-E. Wang, Nilpotent dessins: Decomposition theorem and classification of the abelian dessins, preprint.
[8] I.M. Isaacs, Finite Group Theory, American Mathematical Society, Providence RI, 2008.
[9] G.A. Jones, Elementary abelian regular coverings of platonic maps, Case I: ordinary representations, J. Algebraic Combin. 41 (2014) 461–491.
[10] G.A. Jones, D. Singerman, Theory of maps on orientable surfaces, Proc. Lond. Math. Soc. (3) 37 (1978) 273–307.
[11] G.A. Jones, D.B. Surowski, Cyclic regular coverings of the platonic maps, European J. Combin. 21 (2000) 333–345.
[12] C.H. Li, J. Širáň, Regular maps whose groups do not act faithfully on vertices, edges, or faces, European J. Combin. 26 (2005) 521–541.
[13] A. Malnič, R. Nedela, M. Škoviera, Regular homomorphisms and regular maps, European J. Combin. 23 (2002) 449–461.
[14] G.A. Miller, Generalization of the groups of genus zero, Trans. Amer. Math. Soc. 8 (1907) 1–13.
[15] W.C. Schroeder, D.B. Surowski, Regular cyclic coverings of regular affine maps, European J. Combin. 24 (2003) 1045–1080.
[16] F.A. Sherk, The regular maps on a surface of genus three, Canad. J. Math. 11 (1959) 452–480.
[17] J. Širáň, Coverings of graphs and maps, orthogonality, and eigenvectors, J. Algebraic Combin. 14 (2001) 57–72.
[18] J. Širáň, How Symmetric can Maps on Surfaces be? Surveys in Combinatorics 2013, in: London Math. Soc. Lecture Note Ser., vol. 409, Cambridge Univ. Press, Cambridge, 2013, pp. 161–238.
[19] D.B. Surowski, G.A. Jones, Cohomological constructions of cyclic regular coverings of the platonic maps, European J. Combin. 21 (2000) 407–418.
[20] D.B. Surowski, W.C. Schroeder, Homological methods in combinatorial map theory, European J. Combin. 24 (2003) 1003–1044.